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Reflection Positivity of the Random-Cluster Measure Invalidated for Noninteger q

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We consider the random-cluster Potts measure on a lattice torus that weights each connected component by a positive number q. We show, by constructing a counterexample, that this measure is not reflection-positive unless q is integer.

KEY WORDS: Random-cluster measure; Potts model; reflection positivity.

1. INTRODUCTION

Reflection positivity (RP) has long been known to set up a framework for establishing discontinuous phase transitions in lattice systems.(8, 9, 7, 10, 11, 3) As has been shown recently, in the context of models allowing for a graphical representation, its combination with the latter can substantially reduce the length of the proofs of phase coexistence.(1, 4) In the two-dimensional q-state Potts model, this technique also offers an easy way to establish that the transition occurs exactly at the self-dual point(4) and to improve the method-required bound on q.(2)

The graphical equivalent of the q-state Potts model is the random-cluster measure(6) (RCM). If we refrain from discussing boundary conditions, RCM is a Bernoulli bond-process modified by assigning the number q to each connected component. Since q is not stipulated to be integer in this definition, RCM enables one to think of "extending" the q-state Potts model to non-integer spin-numbers. Similarly, such "extensions" turn out to exist also for various alterations of the Potts model, see, e.g., ref. 1.

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It would be plausible to expect that the above technology based on merging RP with graphical representations encompasses also the continuous "extensions" of the Potts models. Apparently, this is not the case, mainly for the lack of RP—the existing proofs go through the Edwards–Sokal coupling[5] back to the q-state Potts model and, consequently, demand that q be integer. For continuous q, one has been left only with speculations (e.g., about a mapping onto the six-vertex model[2]), so far, that might yield a direct proof of RP at least in some limited domains of q.

In this paper we demonstrate that, in fact, RP does not hold in RCM when q is not integer. It should be emphasized that this does not disqualify the order/disorder phase transition in RCM, since the latter is easily inferred from the discrete case by monotonicity in q. The counterexample we construct involves functions that are highly non-local. Therefore, neither the possibility that RP can be recovered in infinite volume for, e.g., local functions is entirely ruled out.

The rest of the paper is organized as follows. In the next section we give a definition of reflection positivity and state the main result. The proof comes in the third section. The main tool is a suitable representation derived in Proposition, that we believe is of some own interest. In the proof, we are predominantly concerned with RP w.r.t. hyperplanes containing sites. However, as is commented on in the end, the case of RP w.r.t. hyperplanes intersecting bonds is fairly analogous.

2. DEFINITIONS AND MAIN RESULT

Let $\mathscr{T}_N$ be a lattice torus of linear size N, with N being an even integer. We use $\mathbb{B}(\mathscr{T}_N)$ to denote the set of bonds of $\mathscr{T}_N$. Let each bond $b \in \mathbb{B}(\mathscr{T}_N)$ be assigned a variable ω_b taking on values 0 or 1. Let $p \in [0, 1]$ and $q > 0$. Then the *random-cluster measure* (RCM) on torus $\mathscr{T}_N$ with parameters p and q is a probability measure on $\{0, 1\}^{\mathbb{B}(\mathscr{T}_N)}$ weighting the configuration $\boldsymbol{\omega} = (\omega_b)_{b \in \mathbb{B}(\mathscr{T}_N)}$ by the expression

$$\mathbb{P}_{p,q}(\boldsymbol{\omega}) = \frac{1}{Z_N}\, p^{|N_1(\boldsymbol{\omega})|}(1-p)^{|N_0(\boldsymbol{\omega})|}\, q^{C(\boldsymbol{\omega})} \tag{1}$$

Here $N_1(\boldsymbol{\omega}) = \{b\colon \omega_b = 1\}$, $N_0(\boldsymbol{\omega}) = \mathbb{B}(\mathscr{T}_N) \setminus N_1(\boldsymbol{\omega})$, and $C(\boldsymbol{\omega})$ is the number of all connected components that arise from $\mathscr{T}_N$ after cutting all bonds from $N_0(\boldsymbol{\omega})$ (thus, $C(\boldsymbol{\omega})$ includes also the isolated sites). The partition function Z_N provides the appropriate normalization. The expectation w.r.t. $\mathbb{P}_{p,q}$ we denote by $\mathbb{E}$.

Let $P \subset \mathscr{T}_N$ be a hyperplane containing sites, orthogonal to one of the coordinate directions (we think of P as composed of two antipodal components). We use the symbol ϑ_P to denote the reflection w.r.t. P. The components of P divide $\mathcal{T}_N$ into two connected parts $\mathcal{T}_N^{\mathcal{L}}$ and $\mathcal{T}_N^{\mathcal{R}}$, such that $P = \mathcal{T}_N^{\mathcal{L}} \cap \mathcal{T}_N^{\mathcal{R}}$ and $\vartheta_P \mathcal{T}_N^{\mathcal{L}} = \mathcal{T}_N^{\mathcal{R}}$. We use $\mathcal{F}_{\mathcal{L}}(\mathcal{F}_R)$ to denote the set of observables (functions) depending only on $\omega|_{\mathbb{B}(\mathcal{T}_N^{\mathcal{L}})}$ $(\omega|_{\mathbb{B}(\mathcal{T}_N^{\mathcal{R}})})$, respectively), where $\mathbb{B}(A)$ stands for the set of bonds whose both ends lie in $A \subset \mathcal{T}_N$.

Definition. Suppose a measure $\tilde{\mathbb{P}}$ acting on ω be given, with the expectation $\tilde{\mathbb{E}}$. We say that $\tilde{\mathbb{P}}$ is *reflection-positive* iff for every $f, g \in \mathcal{F}_{\mathcal{L}}$

$$\tilde{\mathbb{E}}(f \vartheta_P g) = \tilde{\mathbb{E}}(g \vartheta_P f) \tag{2}$$

$$\tilde{\mathbb{E}}(f \vartheta_P f) \geqslant 0 \tag{3}$$

The condition (2) typically follows directly from the symmetry of $\tilde{\mathbb{P}}$ w.r.t. the reflection ϑ_P, so the difficult part to check is (3) (hence also the name). Now we can state our result:

Theorem. Let $q \leqslant N^{d-1}$ and $p \in (0, 1)$. Then the random cluster measure $\mathbb{P}_{p,q}$ on $\mathcal{T}_N$ is reflection-positive if and only if q is a positive integer.

Remark. The singular case of $p = 0$ or 1 leads to $\mathbb{P}_{p,q}$ supported on a single (symmetric) configuration. It follows that $\mathbb{P}_{0,q}$ and $\mathbb{P}_{1,q}$ are reflection-positive for all $q > 0$.

3. PROOF

Fix a hyperplane $P \subset \mathcal{T}_N$. The main idea of the proof is to classify the configurations in $\mathbb{B}(\mathcal{T}_N^{\mathcal{L}})$ according to the partitions of P into connected sets induced thereby. In this way a representation of the l.h.s. of (3) can be derived that bids an opportunity to choose f so that the reflection positivity of RCM is violated.

Let $\omega_{\mathcal{L}}$ be a configuration in $\mathbb{B}(\mathcal{T}_N^{\mathcal{L}}) \backslash \mathbb{B}(P)$, ω_P a configuration in $\mathbb{B}(P)$, and let us use the symbol $\omega_{\mathcal{L}} \vee \omega_P$ to denote the corresponding joint configuration in $\mathbb{B}(\mathcal{T}_N^{\mathcal{L}})$. Then there is a one-to-one correspondence between the partitions of P into sets whose elements are mutually connected via $N_1(\omega_{\mathcal{L}} \vee \omega_P)$, and the graphs G on P whose components are complete graphs. Namely, an edge $(i, j) \in G$ iff i and j $(\in P)$ are connected (i.e., absence of the edge means that the sites are disconnected). We use χ_G to indicate configurations giving rise to G. Let

$$\tilde{W}_{\mathcal{L}}(\omega_{\mathcal{L}} \vee \omega_P) = \frac{1}{\sqrt{Z_N}} p^{|N_1(\omega_{\mathcal{L}})| + |N_1(\omega_P)|/2} (1-p)^{|N_0(\omega_{\mathcal{L}})| + |N_0(\omega_P)|/2} q^{C_{\mathcal{L}}(\omega_{\mathcal{L}})} \tag{4}$$

where $C_{\mathscr{L}}(\omega_{\mathscr{L}})$ denotes the number of the connected components of $\omega_{\mathscr{L}}$ disconnected from the set $N_1(\omega_P)$. Then we have the following representation of (3):

Proposition. For each $f \in \mathscr{F}_{\mathscr{L}}$, G, and ω_P let $f_G(\omega_P) = \sum_{\omega_{\mathscr{L}}} [f\chi_G \tilde{W}_{\mathscr{L}}](\omega_{\mathscr{L}} \vee \omega_P)$. Then

$$\mathbb{E}(f\vartheta_P f) = \sum_{\omega_P} \sum_G q(q-1)\cdots(q-\imath(G)+1)\left[\sum_{G' \subseteq G} f_{G'}(\omega_P)\right]^2 \tag{5}$$

where $\imath(G)$ denotes the number of components of G.

Proof. We begin with q integer and then use analytic continuation. For integer q, one can devise a coupling[5] between RCM and the Potts model with q spins and $e^{-\beta J} = 1 - p$. Namely, the measure

$$\mathbb{P}_{\mathrm{ES}}(\omega, \sigma) = \frac{1}{Z_N} p^{|N_1(\omega)|}(1-p)^{|N_0(\omega)|} \prod_{\langle i, j\rangle \in N_1(\omega)} \delta_{\sigma_i, \sigma_j} \tag{6}$$

(where $\langle \cdot, \cdot \rangle$ denotes nearest-neighbour sites) has RCM of (1) as its ω-marginal, whereas the σ-marginal is easily identified with the Potts model at the above inverse temperature. If we set

$$W_{\mathscr{L}}(\omega_{\mathscr{L}} \vee \omega_P, \sigma_{\mathscr{L}} \vee \sigma_P) = \frac{1}{\sqrt{Z_N}} p^{|N_1(\omega_{\mathscr{L}})|}(1-p)^{|N_0(\omega_{\mathscr{L}})|} \prod_{\langle i, j\rangle \in N_1(\omega_{\mathscr{L}})} \delta_{\sigma_i, \sigma_j} \times p^{|N_1(\omega_P)|/2}(1-p)^{|N_0(\omega_P)|/2} \tag{7}$$

then the Edwards-Sokal measure allows us to represent $\mathbb{E}(f\vartheta_P f)$ as follows

$$\mathbb{E}(f\vartheta_P f) = \sum_{\omega_P, \sigma_P} \Delta(\omega_P, \sigma_P)\left[\sum_{\omega_{\mathscr{L}}} f(\omega_{\mathscr{L}} \vee \omega_P) \sum_{\sigma_{\mathscr{L}}} W_{\mathscr{L}}(\omega_{\mathscr{L}} \vee \omega_P, \sigma_{\mathscr{L}} \vee \sigma_P)\right]^2 \tag{8}$$

where the function

$$\Delta(\omega_P, \sigma_P) = \prod_{\langle i, j\rangle \in N_1(\omega_P)} \delta_{\sigma_i, \sigma_j} \tag{9}$$

guards that σ_P stays consistent with ω_P.

Now we would like to get the above graphs into play. This is done by observing that unity can be written as

$$\prod_{\substack{(i,j)\\ i,j\in P}} [\delta_{\sigma_i,\sigma_j} + (1-\delta_{\sigma_i,\sigma_j})] = \sum_G \prod_{(i,j)\in G} \delta_{\sigma_i,\sigma_j} \prod_{(i,j)\notin G} (1-\delta_{\sigma_i,\sigma_j}) \tag{10}$$

where the summation goes over all partitions of P (i.e., graphs on P whose connected components are complete graphs) and where the products run over unordered pairs of sites of P.

We insert this expression in (8), just before the square bracket. If $\boldsymbol{\omega}_P$ and $\boldsymbol{\sigma}_P$ are such that the product of Δ- and δ-factors equals one, then two important consequences can be drawn for the summations inside the brackets: first, the graph corresponding to $\boldsymbol{\omega}_{\mathscr{L}} \vee \boldsymbol{\omega}_P$ must be a subgraph of G (sites of P with different spins must not be connected). Second, if also the latter holds, then the summation over $\boldsymbol{\sigma}_{\mathscr{L}}$ yields a number independent of $\boldsymbol{\sigma}_P$. This number is easily identified with $\tilde{W}_{\mathscr{L}}$ from (4).

With these findings, (8) can be rewritten as

$$\begin{aligned}\mathbb{E}(f\vartheta_P f) = \sum_{\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P} \sum_G \Delta(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P) \prod_{(i,j)\in G} \delta_{\sigma_i,\sigma_j} \prod_{(i,j)\notin G} (1-\delta_{\sigma_i,\sigma_j})\\ \times\left[\sum_{\boldsymbol{\omega}_{\mathscr{L}}} f(\boldsymbol{\omega}_{\mathscr{L}} \vee \boldsymbol{\omega}_P)\, \tilde{W}_{\mathscr{L}}(\boldsymbol{\omega}_{\mathscr{L}} \vee \boldsymbol{\omega}_P) \sum_{G'\subseteq G} \chi_{G'}(\boldsymbol{\omega}_{\mathscr{L}} \vee \boldsymbol{\omega}_p)\right]^2\end{aligned} \tag{11}$$

Note that the very last sum gives exactly one whenever $\boldsymbol{\omega}_{\mathscr{L}} \vee \boldsymbol{\omega}_P$ is consistent with G. Now there is nothing to constrain the summation over $\boldsymbol{\sigma}_P$ any more (note that also $\delta(\boldsymbol{\omega}_P, \boldsymbol{\sigma}_P) = 1$ automatically for $\boldsymbol{\omega}_P, \boldsymbol{\sigma}_P$ consistent with G), which gives us the desired claim for all q integer.

For q non-integer, we use continuation in q. First observe that (5) is expressed purely in terms of RCM. Then multiplying both sides by Z_N, we recover an equality between polynomials in q. Since (5) holds for all positive integers, we conclude that it holds for all q real (in fact, even q complex, by continuity). ∎

Proof of Theorem. The integer case is a direct consequence of (5). For q non-integer, we describe a counterexample. Let us choose $\boldsymbol{\omega}_P$ such that the set of its connected components $C(\boldsymbol{\omega}_P)$ is large enough. Let us set

$$f = \delta_{\boldsymbol{\omega}_P} \sum_G a_G \chi_G \tag{12}$$

where $\delta_{\boldsymbol{\omega}_P}$ induces $\boldsymbol{\omega}_P$ on $\mathbb{B}(P)$ and the sum restricts to such graphs G that $b_G = \sum_{\boldsymbol{\omega}_{\mathscr{L}}} [\chi_G \tilde{W}_{\mathscr{L}}](\boldsymbol{\omega}_{\mathscr{L}} \vee \boldsymbol{\omega}_P) > 0$. We gather the latter graphs in the set $\mathscr{G}$. Note that $\mathscr{G} \neq \emptyset$, since the *minimal* graph $\bar{G}$, exhibiting only the connections within $\boldsymbol{\omega}_P$, always belongs to $\mathscr{G}$ (as $p \neq 0, 1$, the configuration $\boldsymbol{\omega}_{\mathscr{L}} \vee \boldsymbol{\omega}_P$ with $N_1(\boldsymbol{\omega}_{\mathscr{L}}) = \emptyset$ gets always some non-zero weight under $\tilde{W}_{\mathscr{L}}$).

We shall show that the numbers a_G can be chosen so that the system of $|\mathscr{G}|$ linear equations

$$\sum_{\substack{G, G' \in \mathscr{G} \\ G' \subseteq G}} a_{G'} b_{G'} = \Delta_{\bar{G}}(G) \tag{13}$$

where $\Delta_{\bar{G}}(\cdot)$ indicates $\bar{G}$, are satisfied. For that let us introduce a *linear* order $\prec$ on $\mathscr{G}$, respecting inclusions (i.e., $G \subseteq G'$ implies $G \prec G'$). Such a linear order always exists, as can be easily proved by induction. In the basis labelled according to $\prec$, the l.h.s. of (13) is clearly represented by the *lower-triangular* matrix $B_{G, G'} = b_{G'} 1_{\{G' \subseteq G\}}$, with all the diagonal entries non-vanishing. Consequently, $B_{G, G'}$ is invertible and (13) can be solved in favour of a non-trivial $(a_G)_{G \in \mathscr{G}}$.

Now it remains to convince oneself that the formulas (12, 13) imply that

$$\mathbb{E}(f \vartheta_P f) = q(q-1) \cdots (q - |C(\boldsymbol{\omega}_P)| + 1) \tag{14}$$

which boils down to checking that only the term with $G = \bar{G}$ contributes to the sum over G in (5). If we now choose $\boldsymbol{\omega}_P$ such that $q + 2 > |C(\boldsymbol{\omega}_P)| > q + 1$, then the expression (14) is blatantly negative, since only the last term $(q - |C(\boldsymbol{\omega}_P)| + 1) < 0$. Hence, (3) does not hold in general, unless q is integer. ∎

Remark. The proof we just gave deals with RP w.r.t. hyperplanes on sites. The argument is readily adapted also to the case of hyperplanes intersecting bonds. Namely, in order to derive (5) one has to mind only the following: first, $\boldsymbol{\omega}_P$ is the restriction of $\boldsymbol{\omega}$ to the bonds that intersect P, second, the connectedness issues handled above by χ_G concern now the bonds from $N_1(\boldsymbol{\omega}_P)$, and third, it is the spins congruent with these bonds that are responsible for the crucial prefactor in (5). Once the relation (5) is established, the formulas (12-14) can be taken over almost literally to obtain the desired result.

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